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Variations on Weyl's theorem

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Abstract

In this note we study the property (w), a variant of Weyl's theorem introduced by Rakočević, by means of the localized single-valued extension property (SVEP). We establish for a bounded linear operator defined on a Banach space several sufficient and necessary conditions for which property (w) holds. We also relate this property with Weyl's theorem and with another variant of it, a-Weyl's theorem. We show that Weyl's theorem, a-Weyl's theorem and property (w) for T (respectively T^*) coincide whenever T^* (respectively T) satisfies SVEP. As a consequence of these results, we obtain that several classes of commonly considered operators have property (w).

Keywords: Localized SVEP; Weyl's theorems; Browder's theorems; Property (w)

1. Definitions and basic results

Throughout this paper, X denotes an infinite-dimensional complex Banach space, $L(X)$ the algebra of all bounded linear operators on X. For an operator $T \in L(X)$ we shall denote by $\alpha(T)$ the dimension of the kernel $\ker T$, and by $\beta(T)$ the codimension of the range $T(X)$. Let

$$\Phi_+(X) := \{T \in L(X) \colon \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$$

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be the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(X) := \{T \in L(X)\colon\ \beta(T) < \infty\}$$

be the class of all *lower semi-Fredholm* operators. The class of all semi-Fredholm operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, while the class of all Fredholm operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. If $T \in \Phi_\pm(X)$, the *index* of T is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. Recall that a bounded operator T is said *bounded below* if it injective and has closed range. Evidently, if T is bounded below then $T \in \Phi_+(X)$ and $\operatorname{ind}(T) \leqslant 0$. Define

$$W_+(X) := \{T \in \Phi_+(X)\colon\ \operatorname{ind} T \leqslant 0\},$$

and

$$W_-(X) := \{T \in \Phi_-(X)\colon\ \operatorname{ind} T \geqslant 0\}.$$

The set of *Weyl operators* is defined by

$$W(X) := W_+(X) \cap W_-(X) = \{T \in \Phi(X)\colon\ \operatorname{ind} T = 0\}.$$

The classes of operators defined above generate the following spectra. Denote by

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not bounded below}\}$$

the *approximate point spectrum*, and by

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not surjective}\}$$

the *surjectivity spectrum* of $T \in L(X)$. The *Weyl spectrum* is defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda I - T \notin W(X)\},$$

the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda I - T \notin W_+(X)\},$$

while the *Weyl essential surjectivity spectrum* is defined by

$$\sigma_{\mathrm{lw}}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda I - T \notin W_-(X)\}.$$

Obviously, $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{uw}}(T) \cup \sigma_{\mathrm{lw}}(T)$ and from basic Fredholm theory we have

$$\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ws}}(T^*), \qquad \sigma_{\mathrm{ws}}(T) = \sigma_{\mathrm{uw}}(T^*).$$

Note that $\sigma_{\mathrm{uw}}(T)$ is the intersection of all approximate point spectra $\sigma_{\mathrm{a}}(T + K)$ of compact perturbations K of T, while $\sigma_{\mathrm{lw}}(T)$ is the intersection of all surjectivity spectra $\sigma_{\mathrm{s}}(T + K)$ of compact perturbations K of T, see, for instance, [1, Theorem 3.65].

Let $p := p(T)$ be the *ascent* of an operator T; i.e., the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist we put $p(T) = \infty$. Analogously, let $q := q(T)$ be *descent* of an operator T; i.e., the smallest nonnegative integer q such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ [17, Proposition 38.3]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T, see Heuser [17, Proposition 50.2]. The class of all *upper semi-Browder operators* is defined by

$$B_+(X) := \{T \in \Phi_+(X)\colon\ p(T) < \infty\},$$

while the class of all *lower semi-Browder operators* is defined by

$$B_-(X) := \{T \in \Phi_-(X)\colon\ q(T) < \infty\}.$$

The class of all *Browder operators* is defined by

$$B(X) := B_+(X) \cap B_-(X) = \{T \in \Phi(X):\ p(T),\ q(T) < \infty\}.$$

We have

$$B(X) \subseteq W(X), \qquad B_+(X) \subseteq W_+(X), \qquad B_-(X) \subseteq W_-(X),$$

see [1, Theorem 3.4].

The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \notin B(X)\},$$

the *upper Browder spectrum* is defined by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \notin B_+(X)\},$$

and analogously the *lower Browder spectrum* is defined by

$$\sigma_{\mathrm{lb}}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \notin B_-(X)\}.$$

Clearly, $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T) \cup \sigma_{\mathrm{lb}}(T)$ and $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$.

The single valued extension property plays an important role in local spectral theory, see the recent monograph of Laursen and Neumann [18] and Aiena [1]. In this article we shall consider the following local version of this property, which has been studied in recent papers, [5–7] and previously by Finch [14].

Definition 1.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Clearly, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic function it easily follows that $T \in L(X)$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, T has SVEP at every isolated point of the spectrum $\sigma(T)$. An important subspace in local spectral theory is given by the *glocal spectral subspace* $\mathcal{X}_T(F)$ associated with a closed subset $F \subseteq \mathbb{C}$. This is defined, for an arbitrary operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies the identity $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$.

The basic role of SVEP arises in local spectral theory since all decomposable operators enjoy this property. Recall $T \in L(X)$ has the *decomposition property* (δ) if $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. Decomposable operators may be defined in several ways for instance as the union of the property (β) and the property (δ), see [18, Theorem 2.5.19] for relevant definitions. Note that the property (β) implies that T has SVEP, while the property (δ) implies SVEP for $T^\star$, see [18, Theorem 2.5.19]. Every *generalized scalar* operator on a Banach space is decomposable, see [18] for relevant definitions and results. In particular, every *spectral operators of finite type* is decomposable [11, Theorem 3.6]. Also every operator $T \in L(X)$ with totally disconnected spectrum is decomposable [18, Proposition 1.4.5].

Note that

$$p(\lambda I - T) < \infty \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \tag{1}$$

and dually

$$q(\lambda I - T) < \infty \quad \Rightarrow \quad T^* \text{ has SVEP at } \lambda, \tag{2}$$

see [1, Theorem 3.8]. Furthermore, from definition of SVEP we have

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \tag{3}$$

and dually

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \quad \Rightarrow \quad T^* \text{ has SVEP at } \lambda. \tag{4}$$

Remark 1.2. It should be noted that all the implications above become equivalences if we assume that $\lambda I - T \in \Phi_{\pm}(X)$, see [5,7].

An important subspace in local spectral theory is the *quasi-nilpotent part* of T defined by

$$H_0(T) := \left\{x \in X\colon \lim_{n\to\infty} \left\|T^n x\right\|^{1/n} = 0\right\}.$$

We also have

$$H_0(\lambda I - T) \text{ closed} \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \tag{5}$$

and also this implication is actually an equivalence if $\lambda I - T \in \Phi_{\pm}(X)$, see [5].

Theorem 1.3. [1] *Suppose that $\lambda I - T \in \Phi_{\pm}(X)$. If T has SVEP at λ then* $\operatorname{ind}(\lambda - T) \leqslant 0$*, while if T^* has SVEP at λ then* $\operatorname{ind}(\lambda I - T) \geqslant 0$.

Proof. If $\lambda I - T \in \Phi_{\pm}(X)$ the SVEP for T at λ is equivalent to saying that $p(\lambda I - T) < \infty$ and this implies $\operatorname{ind}(\lambda - T) \leqslant 0$, see [1, Theorem 3.4]. Analogously, the SVEP for T^* at λ is equivalent to saying that $q(\lambda I - T) < \infty$ and this implies $\operatorname{ind}(\lambda - T) \geqslant 0$. □

2. Weyl's theorems and property (w)

Let write $\operatorname{iso} K$ for the set of all isolated points of $K \subseteq \mathbb{C}$. For a bounded operator $T \in L(X)$ set

$$p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \left\{\lambda \in \sigma(T)\colon \lambda I - T \in B(X)\right\}.$$

Note that every $\lambda \in p_{00}(T)$ is a pole of the resolvent and hence an isolated point of $\sigma(T)$, see [17, Proposition 50.2]. Moreover, $p_{00}(T) = p_{00}(T^*)$. Write $\operatorname{iso} K$ for the set of all isolated points of $K \subseteq \mathbb{C}$, and define

$$\pi_{00}(T) := \left\{\lambda \in \operatorname{iso} \sigma(T)\colon 0 < \alpha(\lambda I - T) < \infty\right\}.$$

Obviously,

$$p_{00}(T) \subseteq \pi_{00}(T) \quad \text{for every } T \in L(X). \tag{6}$$

For a bounded operator $T \in L(X)$ let us define

$$\pi_{00}^{a}(T) := \left\{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T)\colon 0 < \alpha(\lambda I - T) < \infty\right\},$$

and

$$p_{00}^{a}(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = \left\{\lambda \in \sigma_{\mathrm{a}}(T)\colon \lambda I - T \in \mathcal{B}_{+}(X)\right\}.$$

Lemma 2.1. *For every $T \in L(X)$ we have*

$$p_{00}(T) \subseteq p_{00}^{a}(T) \subseteq \pi_{00}^{a}(T) \quad \textit{and} \quad \pi_{00}(T) \subseteq \pi_{00}^{a}(T). \tag{7}$$

Proof. If $\lambda \in p_{00}(T)$ then λ is an isolated point of $\sigma(T)$. Moreover, $\lambda \in \sigma_{\mathrm{a}}(T)$ since $\alpha(\lambda I - T) > 0$ (in fact, if were $\alpha(\lambda I - T) = 0$, we would have $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$, see [1, Theorem 3.4(iii)], and hence $\lambda \notin \sigma(T)$, a contradiction). Therefore λ is an isolated point of $\sigma_{\mathrm{a}}(T)$, so the inclusion $p_{00}(T) \subseteq p_{00}^{a}(T)$ is proved.

To show the inclusion $p_{00}^{a}(T) \subseteq \pi_{00}^{a}(T)$, let $\lambda \in p_{00}^{a}(T)$. Then $\lambda I - T \in \Phi_{+}(X)$ and $p(\lambda I - T) < \infty$. According Remark 1.2 then λ is isolated in $\sigma_{\mathrm{a}}(T)$. Furthermore, $0 < \alpha(\lambda I - T) < \infty$ since $(\lambda I - T)(X)$ is closed and $\lambda \in \sigma_{\mathrm{a}}(T)$. The inclusion $\pi_{00}(T) \subseteq \pi_{00}^{a}(T)$ is clear. □

Following Harte and W.Y. Lee [16], we shall say that T satisfies *Browder's theorem* if

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T),$$

while, $T \in L(X)$ is said to satisfy *a-Browder's theorem* if

$$\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T).$$

Browder's theorem and a-Browder's theorem may be characterized by localized SVEP in the following way:

Theorem 2.2. [3,4] *If $T \in L(X)$ the following equivalences hold*:

(i) *T satisfies Browder's theorem $\Leftrightarrow$ T has SVEP at every $\lambda \notin \sigma_{\mathrm{w}}(T)$;*
(ii) *T satisfies a-Browder's theorem $\Leftrightarrow$ T has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$.*

Moreover, the following statements hold:

(iii) *If T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ then a-Browder's theorem holds for T^*.*
(iv) *If T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$ then a-Browder's theorem holds for T.*

Obviously,

$$a\text{-Browder's theorem holds for } T \quad \Rightarrow \quad \text{Browder's theorem holds for } T$$

and the converse is not true.

Remark 2.3. The opposite implications of (iii) and (iv) in Theorem 2.2 in general do not hold. In [2] it is given an example of unilateral weighted left shift on $\ell^q(\mathbb{N})$ which shows that these implications cannot be reversed.

By Theorem 2.2 we also have

$$T \text{ or } T^* \text{ has SVEP} \quad \Rightarrow \quad a\text{-Browder's theorem holds for both } T, T^*.$$

Following Coburn [10], we say that *Weyl's theorem holds* for $T \in L(X)$ if

$$\Delta(T) := \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T). \tag{8}$$

An approximate point version of Weyl's theorem is a-Weyl's theorem: according Rakočević [23] an operator $T \in L(X)$ is said to satisfy a-Weyl's theorem if

$$\Delta_a(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^a(T).$$

Since $\lambda I - T \in W_+(X)$ implies that $(\lambda I - T)(X)$ is closed, we can write

$$\Delta_a(T) = \{\lambda \in \mathbb{C}\colon \lambda I - T \in W_+(X),\ 0 < \alpha(\lambda I - T)\}.$$

It should be noted that the set $\Delta_a(T)$ may be empty. This is, for instance, the case of a right shift on $\ell^2(\mathbb{N})$, see [4]. Furthermore,

$$a\text{-Weyl's theorem holds for } T \quad \Rightarrow \quad \text{Weyl's theorem holds for } T,$$

while the converse in general does not hold.

Theorem 2.4. [2] *Suppose that $T \in L(X)$. Then the following assertions hold*:

(i) *Weyl's theorem holds for T if and only if Browder's theorem holds for T, or equivalently for T^*, and $\pi_{00}(T) = p_{00}(T)$.*
(ii) *a-Weyl's theorem holds for T if and only if a-Browder's theorem holds for T and $p_{00}^a(T) = \pi_{00}^a(T)$.*

The following variant of Weyl's theorem has been introduced by Rakočević [22].

Definition 2.5. A bounded operator $T \in L(X)$ is said to satisfy property (w) if

$$\Delta_a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T).$$

Unlike a-Weyl's theorem, the study of property (w) has been rather neglected, although, exactly like a-Weyl's theorem, property (w) implies Weyl's theorem (see next Theorem 2.8). In the present article we shall study this property and give several characterizations of it by using the localized SVEP. In particular we shall relate property (w) to Weyl's theorem, and with to a-Weyl's theorem in the case that T or its dual T^* has SVEP.

The first result shows that property (w) entails a-Browder's theorem.

Theorem 2.6. *Suppose that $T \in L(X)$ satisfies property (w). Then a-Browder's holds for T and*

$$\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{uw}}(T) \cup \operatorname{iso} \sigma_{\mathrm{a}}(T).$$

Proof. By part (ii) of Theorem 2.2 it suffices to show that T has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$. Let $\lambda \notin \sigma_{\mathrm{uw}}(T)$. If $\lambda \notin \sigma_{\mathrm{a}}(T)$ then T has SVEP at λ by (3), while if $\lambda \in \sigma_{\mathrm{a}}(T)$, then $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$ and hence $\lambda \in \operatorname{iso} \sigma(T)$, so also in this case T has SVEP at λ.

The inclusion $\sigma_{\mathrm{uw}}(T) \cup \operatorname{iso} \sigma_{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{a}}(T)$ holds for every $T \in L(X)$, since $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{a}}(T)$. To show the opposite implication, suppose that T satisfies property (w) and $\lambda \in \sigma_{\mathrm{a}}(T)$. If $\lambda \notin \sigma_{\mathrm{uw}}(T)$ then $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$ and hence $\lambda \in \operatorname{iso} \sigma(T)$, in particular $\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T)$, so $\sigma_{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{uw}}(T) \cup \operatorname{iso} \sigma_{\mathrm{a}}(T)$, as desired. $\square$

Property (w) may be characterized in the following way:

Theorem 2.7. *If $T \in L(X)$ the following statements are equivalent*:

(i) *T satisfies property (w)*;
(ii) *a-Browder's theorem holds for T and $p^a_{00}(T) = \pi_{00}(T)$.*

Proof. (i) $\Rightarrow$ (ii) By Theorem 2.6 we need only to prove the equality $p^a_{00}(T) = \pi_{00}(T)$. If $\lambda \in \pi_{00}(T) = \lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ then $\lambda \in \sigma_{\mathrm{a}}(T)$ and $\lambda I - T \in W_+(X)$. Since λ is isolated in $\sigma(T)$ the SVEP of T at λ is equivalent to saying that $p(\lambda I - T) < \infty$, so $\lambda \in p^a_{00}(T)$. Hence $\pi_{00}(T) \subseteq p^a_{00}(T)$.

To show the opposite inclusion, suppose that $\lambda \in p^a_{00}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T)$. Since by Theorem 2.6 T satisfies a-Browder's theorem then $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T)$, so $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$. Therefore the equality $p^a_{00}(T) = \pi_{00}(T)$ is proved.

(ii) $\Rightarrow$ (i) If $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ then a-Browder's theorem entails that $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p^a_{00}(T) = \pi_{00}(T)$. Conversely, if $\lambda \in \pi_{00}(T)$ then $\lambda \in p^a_{00}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$. Hence $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$. □

As observed by Rakočević [22], property (w) implies Weyl's theorem. We shall give a proof of this result by using SVEP.

Theorem 2.8. *If $T \in L(X)$ satisfies property (w) then Weyl's theorem holds for T.*

Proof. Suppose that T satisfies property (w). By Theorem 2.6 T satisfies a-Browder's theorem and hence Browder's theorem. From part (i) of Theorem 2.4 we need only to prove that $\pi_{00}(T) = p_{00}(T)$. If $\lambda \in \pi_{00}(T)$ then $\lambda \in \sigma_{\mathrm{a}}(T)$, since $\alpha(\lambda I - T) > 0$, and from $\lambda \in \operatorname{iso}\sigma(T)$ we know that both T and T^* have SVEP at λ. From the equality $\pi_{00}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ we see that $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and hence $\lambda I - T \in \Phi_+(X)$. The SVEP for T and T^* at λ by Remark 1.2 implies that $p(\lambda I - T) = q(\lambda I - T) < \infty$. From Theorem 3.4 of [1] we then obtain that $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, so $\lambda \in p_{00}(T)$. Hence $\pi_{00}(T) \subseteq p_{00}(T)$, and since the reverse inclusion holds for every $T \in L(X)$ we conclude that $\pi_{00}(T) = p_{00}(T)$. □

The reverse of the result of Theorem 2.8 generally does not hold, see next Example 2.14. Define

$$\Lambda(T) := \{\lambda \in \Delta_a(T) \colon \operatorname{ind}(\lambda I - T) < 0\}.$$

Clearly,

$$\Delta_a(T) = \Delta(T) \cup \Lambda(T) \quad \text{and} \quad \Lambda(T) \cap \Delta(T) = \emptyset. \tag{9}$$

The next result relates Weyl's theorem and property (w).

Theorem 2.9. *If $T \in L(X)$ satisfies property (w) then $\Lambda(T) = \emptyset$. Moreover, the following statements are equivalent*:

(i) *T satisfies property (w)*;
(ii) *T satisfies Weyl's theorem and $\Lambda(T) = \emptyset$*;
(iii) *T satisfies Weyl's theorem and $\Delta_a(T) \subseteq \operatorname{iso}\sigma(T)$*;
(iv) *T satisfies Weyl's theorem and $\Delta_a(T) \subseteq \partial\sigma(T)$, $\partial\sigma(T)$ the topological boundary of $\sigma(T)$.*

Proof. Suppose that T satisfies property (w). Suppose that $\Lambda(T)$ is nonempty. Let $\lambda \in \Lambda(T)$. Then $\lambda \in \Delta_a(T) = \pi_{00}(T)$, so λ is isolated in $\sigma(T)$ and hence both T and T^* have SVEP at λ.

Since $\lambda I - T \in W_+(T)$, it then follows by Theorem 1.3 that $\mathrm{ind}(\lambda I - T) = 0$, and this contradicts our assumption that $\lambda \in \Lambda(T)$.

(i) $\Leftrightarrow$ (ii) The implication (i) $\Rightarrow$ (ii) is clear from the first part of the proof and from Theorem 2.8. Conversely, from the equality (9) we see that if $\Lambda(T) = \emptyset$ and T satisfies Weyl's theorem then we have $\Delta_\mathrm{a}(T) = \Delta(T) = \pi_{00}(T)$, so property (w) holds.

(iii) $\Rightarrow$ (ii) Suppose that T satisfies Weyl's theorem. If $\Delta_a(T) \subseteq \mathrm{iso}\,\sigma(T)$, then both T and T^* have SVEP at every $\lambda \in \Delta_a(T)$. As in the first part of the proof, this implies that $\mathrm{ind}(\lambda I - T) = 0$ for every $\lambda \in \Delta_a(T)$, so $\Lambda(T) = \emptyset$. Hence property (w) holds for T.

(i) $\Rightarrow$ (iii) If property (w) holds then $\Delta_a(T) = \pi_{00}(T) \subseteq \mathrm{iso}\,\sigma(T)$.

(iii) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (ii) Both T and T^* have SVEP at every point of $\partial\sigma(T) = \partial\sigma(T^*)$, so by Theorem 1.3, $\mathrm{ind}(\lambda I - T) = 0$ for all $\lambda \in \Delta_\mathrm{a}(T)$, and hence $\Lambda(T) = \emptyset$. $\square$

The condition $\Lambda(T) = \emptyset$ is satisfied by every *Riesz operator* $T \in L(X)$ on an infinite-dimensional Banach space X, in particular by every compact operator. It is easily seen that Weyl's theorem holds for every compact operator having an infinite spectrum. However, Weyl's theorem may fail for a compact operator T, for an example see [9].

Corollary 2.10. *Suppose that $T \in L(X)$ is decomposable. Then T satisfies property (w) if and only if T satisfies Weyl's theorem.*

Proof. If T is decomposable then both T and T^* have SVEP. This, by Theorem 1.3 entails that $\lambda I - T$ has index 0 for every $\lambda \in \Delta_\mathrm{a}(T)$, and hence $\Lambda(T) = \emptyset$. The equivalence then follows from Theorem 2.9. $\square$

As a consequence of Corollary 2.10, we have that for a bounded operator $T \in L(X)$ having totally disconnected spectrum then property (w) and Weyl's theorem are equivalent.

A bounded operator $T \in L(X)$ is said to have *property $H(p)$* if for all $\lambda \in \mathbb{C}$ there exists a $p := p(\lambda) \in \mathbb{N}$ such that:

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p.$$

Denote by $\mathcal{H}(\sigma(T))$ the set of all analytic functions defined on a neighborhood of $\sigma(T)$, let $f(T)$ be defined by means of the classical functional calculus. In [21] it has been proved that if $T \in L(X)$ has property $H(p)$ then $f(T)$ and $f(T^*)$ satisfy Weyl's theorem. In [22] it was observed that every normal operator on a Hilbert space has property (w). The following result shows that actually we have much more.

Corollary 2.11. *If $T \in L(X)$ is generalized scalar then property (w) holds for both T and T^*. In particular, property (w) holds for every spectral operator of finite type.*

Proof. Every generalized scalar operator T is decomposable and hence also the dual T^* is decomposable, see [18, Theorem 2.5.3]. Moreover, every generalized scalar operator has property $H(p)$ [21, Example 3], so Weyl's theorem holds for both T and T^*. By Corollary 2.10 it then follows that both T and T^* satisfy property (w). The second statement is clear: every spectral operators of finite type is generalized scalar. $\square$

Example 2.12. Property (w), as well as Weyl's theorem, is not transmitted from T to its dual T^*. To see this, consider the weighted right shift $T \in L(\ell^2(\mathbb{N}))$, defined by

$$T(x_1, x_2, \dots) := \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \dots\right) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then

$$T^*(x_1, x_2, \dots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \dots\right) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Both T and T^* are quasi-nilpotent, and hence are decomposable, T satisfies Weyl's theorem since $\pi_{00}(T) = p_{00}(T)$ and hence T has property (w), by Corollary 2.10. On the other hand, we have $\pi_{00}(T^*) = \{0\} \neq \sigma(T^*) \setminus \sigma_{\mathrm{w}}(T^*) = \emptyset$, so T^* does not satisfy Weyl's theorem. Since T^* is decomposable, by Corollary 2.10 then T^* does not satisfy property (w).

The following examples show that property (w) is not intermediate between Weyl's theorem and a-Weyl's theorem. The first example provides an operator satisfying property (w) but not a-Weyl's theorem.

Example 2.13. Let T be the hyponormal operator T given by the direct sum of the 1-dimensional zero operator and the unilateral right shift R on $\ell^2(\mathbb{N})$. Then $\sigma(T) = \mathbf{D}$, $\mathbf{D}$ the closed unit disc in $\mathbb{C}$. Moreover, 0 is an isolated point of $\sigma_{\mathrm{a}}(T) = \Gamma \cup \{0\}$, Γ the unit circle of $\mathbb{C}$, and $0 \in \pi_{00}^a(T)$, while $0 \notin p_{00}^a(T) = \emptyset$, since $p(T) = p(R) = \infty$. Hence, by Theorem 2.4, T does not satisfy a-Weyl's theorem. On the other hand $\pi_{00}(T) = \emptyset$, since $\sigma(T)$ has no isolated points, so $p_{00}^a(T) = \pi_{00}(T)$. Since every hyponormal operator has SVEP we also know that a-Browder's theorem holds for T, so from Theorem 2.7 we see that property (w) holds for T.

The following example provides an operator that satisfies a-Weyl theorem but not property (w).

Example 2.14. Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and

$$U(x_1, x_2, \dots) := (0, x_2, x_3, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

If $T := R \oplus U$ then $\sigma(T) = \mathbf{D}$ so $\operatorname{iso}\sigma(T) = \pi_{00}(T) = \emptyset$. Moreover, $\sigma_{\mathrm{a}}(T) = \Gamma \cup \{0\}$, $\sigma_{\mathrm{uw}}(T) = \Gamma$, so T does not satisfy property (w), since $\Delta_a(T) = \{0\}$. On the other hand we also have $\pi_{00}^a(T) = \{0\}$, so T satisfies a-Weyl's theorem.

We give now two sufficient conditions for which a-Weyl's theorem for T (respectively T^*) implies property (w) for T (respectively T^*). Observe that these conditions, by the implications (iii) and (iv) of Theorem 2.2 and Remark 2.3, are a bit stronger than the assumption that T satisfies a-Browder's theorem.

Theorem 2.15. *If $T \in L(X)$ the following statements hold*:

(i) *If T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and T satisfies a-Weyl's theorem then property (w) holds for T.*
(ii) *If T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ and T^* satisfies a-Weyl's theorem then property (w) holds for T^*.*

Proof. (i) We show first that if T^* has SVEP at every point $\lambda \notin \sigma_{\mathrm{uw}}(T)$ then $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) \subseteq \pi_{00}(T)$. Let $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$. Since by part (iv) of Theorem 2.2 T satisfies a-Browder's theorem then $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$, so $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p_{00}^a(T) \subseteq \pi_{00}^a(T)$, so $\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T)$.

On the other hand, since $\lambda I - T \in B_+(X)$, by Remark 1.2 we know that the SVEP for T^* at λ implies that $\lambda \in \operatorname{iso} \sigma_{\mathrm{s}}(T)$. Therefore, $\lambda \in \operatorname{iso} \sigma(T)$. Since $\lambda \in \sigma_{\mathrm{a}}(T)$ and $\lambda I - T$ has closed range we also have $0 < \alpha(\lambda I - T) < \infty$, and hence $\lambda \in \pi_{00}(T)$. This shows the inclusion $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) \subseteq \pi_{00}(T)$. To prove the opposite inclusion observe that a-Weyl's theorem for T entails that $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^a(T) \supseteq \pi_{00}(T)$. Hence $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$, so property (w) holds for T.

(ii) Suppose that T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$, and suppose that $\lambda \in \sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{uw}}(T^*)$. By part (iii) of Theorem 2.2 then T^* satisfies a-Browder's theorem, so $\sigma_{\mathrm{uw}}(T^*) = \sigma_{\mathrm{ub}}(T^*)$ and by duality $\sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{lb}}(T)$. Hence $\lambda \in \sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{uw}}(T^*) = \sigma_{\mathrm{s}}(T) \setminus \sigma_{\mathrm{lb}}(T)$. Therefore $\lambda I - T \in B_-(X)$ and hence $q(\lambda I - T) < \infty$. This implies the SVEP for T^* at λ, or equivalently that $\lambda \in \operatorname{iso} \sigma_{\mathrm{s}}(T)$. Since $\lambda I - T \in \Phi_-(X)$, our assumption of SVEP of T at λ entails also that $\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T)$. Hence, $\lambda \in \operatorname{iso} \sigma(T) = \operatorname{iso} \sigma(T^*)$. Furthermore, since $\lambda \in \sigma_{\mathrm{s}}(T)$ and $\lambda I - T$ is semi-Fredholm we have $\alpha(\lambda I - T^*) = \beta(\lambda I - T) > 0$, so $\lambda \in \pi_{00}(T^*)$. This proves the inclusion $\sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{uw}}(T^*) \subseteq \pi_{00}(T^*)$. Finally, a-Weyl's theorem for T^* entails that $\sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{uw}}(T^*) = \pi_{00}^a(T^*) \supseteq \pi_{00}(T^*)$, so $\sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{uw}}(T^*) = \pi_{00}(T^*)$ and hence property (w) holds for T^*. □

The next result shows that Weyl's theorems and property (w) are equivalent in presence of SVEP.

Theorem 2.16. *Let $T \in L(X)$. Then the following equivalences holds*:

(i) *If T^* has SVEP, the property (w) holds for T if and only if Weyl's theorem holds for T, and this is the case if and only if a-Weyl's theorem holds for T.*
(ii) *If T has SVEP, the property (w) holds for T^* if and only if Weyl's theorem holds for T^*, and this is the case if and only if a-Weyl's theorem holds for T^*.*

Proof. (i) By Theorem 2.8 and part (i) of Theorem 2.15, for T we have the implications

$$a\text{-Weyl} \quad \Rightarrow \quad (\omega) \quad \Rightarrow \quad \text{Weyl.} \tag{10}$$

Assume now that T satisfies Weyl's theorem. The SVEP of T^* implies that $\sigma(T) = \sigma_{\mathrm{a}}(T)$, see [1, Corollary 2.5], so $\pi_{00}^a(T) = \pi_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$. Furthermore, by [1, Corollary 3.53] we also have $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{ub}}(T)$ from which it follows that $\pi_{00}^a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p_{00}^a(T)$. Since the SVEP for T^* implies a-Browder's theorem for T we then conclude, by part (ii) of Theorem 2.4, that a-Weyl's theorem holds for T.

(ii) The argument is similar to that used in the proof of part (i). The implication (10) holds for T^* by Theorem 2.8 and part (ii) of Theorem 2.15. If T has SVEP then $\sigma(T^*) = \sigma(T) = \sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{a}}(T^*)$, see [1, Corollary 2.5], and hence $\pi_{00}^a(T^*) = \pi_{00}(T^*)$. Moreover, by [1, Corollary 3.53] we also have

$$\sigma_{\mathrm{w}}(T^*) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ub}}(T^*),$$

from which it easily follows that $\pi_{00}^a(T^*) = p_{00}^a(T^*)$. The SVEP for T implies that T^* satisfies a-Browder's, so by part (ii) of Theorem 2.4 a-Weyl's theorem holds for T^*. □

Remark 2.17. The operator T considered in Example 2.13 shows that in the statement (i) of Theorem 2.16 the SVEP for T^* cannot be replaced by the SVEP for T. Similarly, in the statement

(ii) of Theorem 2.16 we cannot replace the SVEP for T with the SVEP for T^*. For instance, let $0 < \varepsilon < 1$ and define $T \in L(\ell^2(\mathbb{N}))$ by

$$T(x_1, x_2, \ldots) := (\varepsilon x_1, 0, x_2, x_3, \ldots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then $\sigma_{\mathrm{a}}(T^*) = \Gamma \cup \{0\}$ [24], and hence $\operatorname{int} \sigma_{\mathrm{a}}(T^*) = \emptyset$, which implies that T^* has SVEP. Moreover, $\sigma_{\mathrm{uw}}(T^*) = \Gamma$, $\pi_{00}^{a}(T^*) = \{\varepsilon\}$, so a-Weyl's theorem holds for T^*. On the other hand, it is easy to see that $\pi_{00}(T^*) = \emptyset$, so property (w) does not hold for T^*.

Corollary 2.18. *If T is generalized scalar then property (w) holds for both $f(T)$ and $f(T^*)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. Since T has property $H(p)$ then Weyl's theorem holds for $f(T)$ and $f(T^*)$, see [21, Corollary 3.6]. Moreover, T and T^* being decomposable, both T and T^* have SVEP, hence also $f(T)$ and $f(T^*) = f(T)^*$ have SVEP by Theorem 2.40 of [1]. By Theorem 2.16 it then follows that property (w) holds for both $f(T)$ and $f(T^*)$. □

Remark 2.19. Corollary 2.18 applies to a large number of the classes of operators defined in Hilbert spaces. In [21] Oudghiri observed that every *subscalar operator* T (i.e., T is similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces) has property $H(p)$. Consequently, property $H(p)$ is satisfied by p-hyponormal operators and *log*-hyponormal operators [19, Corollary 2], w-hyponormal operators [20], M-hyponormal operators [18, Proposition 2.4.9], and totally paranormal operators [8]. Also totally $*$-paranormal operators have property $H(1)$ [15]. The next corollary shows that if T' belongs to one of the above mentioned classes of operators then property (w) is satisfied by $f(T)$ and $f(T^*)$ for all $f \in \mathcal{H}(\sigma(T))$.

In the case of operators defined on Hilbert spaces instead of the dual T^* it is more appropriate to consider the Hilbert adjoint T' of $T \in L(H)$.

Corollary 2.20. *If T' has property $H(p)$ then property (w) holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$. In particular, if T' is generalized scalar then property (w) holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. If T' has property $H(p)$ then a-Weyl's theorem holds for $f(T)$ [2, Theorem 4.1]. Moreover, T' has SVEP, and, as observed in [2], this entails that also T^* has SVEP. Therefore, $f(T)^* = f(T^*)$ has SVEP so part (i) of Theorem 2.16 applies. □

From Corollary 2.20 it then follows that if T' belongs to each one of the classes of operators mentioned in Remark 2.19 then property (w) holds for $f(T)$.

A similar result holds for algebraically paranormal operators. If T' is algebraically paranormal then, see [1, Theorem 2.40], a-Weyl's theorem holds for $f(T)$. Moreover, T' has SVEP and hence $f(T)^* = f(T^*)$ has SVEP, so by Theorem 2.16 $f(T)$ satisfies property (w) for all $f \in \mathcal{H}(\sigma(T))$.

An operator $T \in L(X)$ is said to be *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent operator $(\lambda I - T)^{-1}$, or equivalently $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$, see [17, Proposition 50.2]. An operator $T \in L(X)$ is said to be *a-polaroid* if every isolated point of $\sigma_{\mathrm{a}}(T)$

is a pole of the resolvent operator $(\lambda I - T)^{-1}$, or equivalently $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$, see [17, Proposition 50.2]. Clearly,

$$T \ a\text{-polaroid} \quad \Rightarrow \quad T \text{ polaroid}$$

and the opposite implication is not generally true.

Theorem 2.21. *Suppose that T is a-polaroid. Then a-Weyl's theorem holds for T if and only if T satisfies property (w).*

Proof. Note first that if T is a-polaroid then $\pi^a_{00}(T) = p_{00}(T)$. In fact, if $\lambda \in \pi^a_{00}(T)$ then λ is isolated in $\sigma_{\mathrm{a}}(T)$ and hence $p(\lambda I - T) = q(\lambda I - T) < \infty$. Moreover, $\alpha(\lambda I - T) < \infty$, so by Theorem 3.4 of [1] it follows that $\beta(\lambda I - T)$ is also finite, thus $\lambda \in p_{00}(T)$. This shows that $\pi^a_{00}(T) \subseteq p_{00}(T)$, and consequently by Lemma 2.1 we have $\pi^a_{00}(T) = p_{00}(T)$.

Now, if T satisfies a-Weyl's theorem then $\Delta_a(T) = \pi^a_{00}(T) = p_{00}(T)$, and since Weyl's theorem holds for T we also have by Theorem 2.4 that $p_{00}(T) = \pi_{00}(T)$. Hence property (w) holds for T.

Conversely, if T satisfies property (w) then $\Delta_a(T) = \pi_{00}(T)$. Since by Theorem 2.8 T satisfies Weyl's theorem we also have, by Theorem 2.4, $\pi_{00}(T) = p_{00}(T) = \pi^a_{00}(T)$, so T satisfies a-Weyl's theorem. □

The last theorem implies that property (w) holds for every multiplier $T \in M(A)$ of a commutative semi-simple regular Tauberian Banach algebra A, and in particular for every convolution operator on $L^1(G)$, where G is a compact Abelian group. In fact a-Weyl's theorem holds for T [8], and by Theorem 5.54 and Theorem 4.36 of [1], T is a-polaroid.

Theorem 2.22. *Suppose that T is a-polaroid and that T^* has SVEP. Then $f(T)$ satisfies property (w) for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. If T is a-polaroid then T is a-isoloid (i.e., every isolated point of $\sigma_a(T)$ is an eigenvalue of T). The SVEP for T^* ensures that the spectral mapping theorem holds for $\sigma_{\mathrm{uw}}(T)$, i.e., if $f \in \mathcal{H}(\sigma(T))$ then $\sigma_{\mathrm{uw}}(f(T)) = f(\sigma_{\mathrm{uw}}(T))$, see [12] or [1, Theorem 3.66]. By Theorem 5.4 of [13] then $f(T)$ satisfies a-Weyl's theorem, and since $f(T^*) = f(T)^*$ has SVEP from Theorem 2.16 we conclude that property (w) holds for $f(T)$. □

Remark 2.23. In the proof of the next theorem we shall use the following basic result. Suppose that for a linear operator T we have $\alpha(T) < \infty$. Then $\alpha(T^n) < \infty$ for all $n \in \mathbb{N}$. To see this we use an inductive argument. Assume that $\dim \ker T^n < \infty$. Since $T(\ker T^{n+1}) \subseteq \ker T^n$ then the restriction $T_0 := T | \ker T^{n+1} : \ker T^{n+1} \to \ker T^n$ has kernel equal to $\ker T$ so the canonical mapping $\hat{T} : \ker T^{n+1} / \ker T \to \ker T^n$ is injective. From this it follows that $\dim \ker T^{n+1} / \ker T \leqslant \ker T^n < \infty$, and since $\dim \ker T < \infty$ we may conclude that $\dim \ker T^{n+1} < \infty$.

The operator defined in Example 2.14 shows that a similar result to that of Theorem 2.21 does not hold for polaroid operators, i.e., if $T \in L(X)$ is polaroid Weyl's theorem for T and property (w) for T in general are not equivalent. However, we have

Theorem 2.24. *Suppose that $T \in L(X)$. Then the following statements hold*:

(i) *If T is polaroid and T has SVEP then property (w) holds for T^*.*
(ii) *If T is polaroid and T^* has SVEP then property (w) holds for T.*

Proof. (i) By Theorem 2.16 it suffices to show that Weyl's theorem holds for T^*. The SVEP ensures that Browder's theorem holds for T^*. We prove that $\pi_{00}(T^*) = p_{00}(T^*)$. Let $\lambda \in \pi_{00}(T^*)$. Then $\lambda \in \operatorname{iso}\sigma(T^*) = \operatorname{iso}\sigma(T)$ and the polaroid assumption implies that λ is a pole of the resolvent, or equivalently $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. If P denotes the spectral projection associated with $\{\lambda\}$ we have $(\lambda I - T)^p(X) = \ker P$ [1, Theorem 3.74], so $(\lambda I - T)^p(X)$ is closed, and hence also $(\lambda I - T^*)^p(X^*)$ is closed. Since $\lambda \in \pi_{00}(T^*)$ then $\alpha(\lambda I^* - T^*) < \infty$ and this implies $\alpha(\lambda I^* - T^*)^p < \infty$, from which we conclude that $(\lambda I^* - T^*)^p \in \Phi_+(X^*)$, hence $\lambda I - T^* \in \Phi_+(X^*)$, and consequently $\lambda I - T \in \Phi_-(X)$. Therefore $\beta(\lambda I - T) < \infty$ and since $p(\lambda I - T) = q(\lambda I - T) < \infty$ by Theorem 3.4 of [1] we then conclude that $\alpha(\lambda I - T) < \infty$. Hence $\lambda \in p_{00}(T) = p_{00}(T^*)$. This proves that $\pi_{00}(T^*) \subseteq p_{00}(T^*)$, and since by Lemma 2.1 the opposite inclusion is satisfied by every operator we may conclude that $\pi_{00}(T^*) = p_{00}(T^*)$. By Theorem 2.4 then T^* satisfies Weyl's theorem.

(ii) The SVEP for T^* implies that Browder's theorem holds for T. Again by Theorem 2.16 it suffices to show that T satisfies Weyl's theorem, and hence by Lemma 2.1 and Theorem 2.4 we need only to prove that $\pi_{00}(T) = p_{00}(T)$. Let $\lambda \in \pi_{00}(T)$. Then $\lambda \in \operatorname{iso}\sigma(T)$ and since T is polaroid then $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. Since $\alpha(\lambda I - T) < \infty$ we then have $\beta(\lambda I - T) < \infty$ and hence $\lambda \in p_{00}(T)$. Hence $\pi_{00}(T) \subseteq p_{00}(T)$ and by Lemma 2.1 we then conclude that $\pi_{00}(T) = p_{00}(T)$. □

Part (i) of Theorem 2.24 shows that the dual T^* of a multiplier $T \in M(A)$ of a commutative semi-simple Banach algebra A has property (w), since every multiplier $T \in M(A)$ of a commutative semi-simple Banach algebra satisfies Weyl's theorem and is polaroid, see [1, Theorem 4.36].

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